

On the middle convolution

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Abstract

In [9], a purely algebraic analogon of Katz' middle convolution functor (see [12]) is given. It is denoted by MC_λ . In this paper, we present a cohomological interpretation of MC_λ and find an explicit Riemann-Hilbert correspondence for this functor. This leads to an algorithm for the construction of Fuchsian systems corresponding to irreducible rigid local systems under the Riemann-Hilbert correspondence. Also, we describe the effect of MC_λ on the p -curvatures and find new examples of differential equations for which the Grothendieck-Katz p -curvature conjecture holds.

1 Introduction

Let D be a complex ordinary differential equation of order n or, equivalently, a linear system of differential equations of rank n . Let $T = \{t_1, \dots, t_r\} \subseteq \mathbb{C}$ denote the set of finite singularities of D and let γ be a closed path in $X = \mathbb{C} \setminus T$. Analytic continuation of a fundamental matrix F of D along γ transforms F into $F \cdot A$, with $A \in \mathrm{GL}_n(\mathbb{C})$ uniquely determined. One calls A the *monodromy matrix* of D with respect to γ . In other words, if \mathcal{F} denotes the local system on X , formed by the solutions of D , then A describes the monodromy of \mathcal{F} along γ .

Since Riemann's investigations on the hypergeometric equations ([16]), the use of monodromy is one of the most powerful tools in the investigation of integrable differential equations.

It is a basic fact, already used by Riemann, that the solutions of the hypergeometric differential equations give rise to a physically rigid local system, see [12], Introduction. This means essentially, that the global behaviour of the solutions under analytic continuation is determined by the local behaviour at the singularities (including ∞).

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A description of all irreducible and physically rigid local systems on the punctured affine line was given by Katz [12]. The main tool herefore is a middle convolution functor on the category of perverse sheaves, loc. cit., Chap. 5. This functor is denoted by MC_χ , for χ a one-dimensional representation of $\pi_1(\mathbb{G}_m)$. It preserves important properties of local systems like the index of rigidity and irreducibility, but in general, MC_χ changes the rank and the monodromy group. As an application, Katz shows that any irreducible rigid local system on the punctured affine line can be obtained from a one-dimensional local system by applying iteratively a suitable sequence of middle convolutions MC_{χ_i} and scalar multiplications, loc. cit., Chap. 6. Since the effect of MC_{χ_i} on the local monodromy can be determined via Laumon's theory of l -adic Fourier transform, this leads to an existence algorithm for rigid local systems, loc. cit., Section 6.4.

In [9], the authors give a purely algebraic analogon of the functor MC_χ (the construction is reviewed in Section 2). This analogous functor is a functor of the category $\text{Mod}(K[F_r])$ of modules of the free group F_r on r generators to itself. It depends on a scalar $\lambda \in \mathbb{C}^\times$ and is denoted by MC_λ .

One has the equivalence between $\text{Mod}(\mathbb{C}[F_r]) \cong \text{Mod}(\mathbb{C}[\pi_1(X, x_0)])$ and the category $\text{LocSys}(X)$ of local systems on X , see Section 4.1. Then, MC_λ translates into a functor of the category of local systems on the r -punctured complex affine line X to itself, sending a local system \mathcal{F} to $MC_\lambda(\mathcal{F})$, see Section 4.2. It follows from the results of [9] that MC_λ (viewed as a functor on the category of local systems on X) has analogous properties as Katz' functor MC_χ , where χ is the representation, sending the standard generator of $\pi_1(\mathbb{G}_m(\mathbb{C}))$ to λ . This leads to a new and elementary proof of Katz' existence algorithm for rigid local systems, see [9], Chap. 4. Similar results are obtained in [19].

It is the aim of this paper to give answers to the following problems:

Problem 1: Give a cohomological interpretation of $MC_\lambda(\mathcal{F})$, explaining the formal similarity between MC_λ and Katz' functor MC_χ .

By the work of Deligne, it is known that the category of complex local systems on $X = \mathbb{C} \setminus T$ is equivalent to the category of ordinary complex differential equations with polynomial coefficients having only regular singularities at the missing points (including ∞) and no singularities in X , see [8]. This equivalence is called the *Riemann-Hilbert correspondence*. This leads to

Problem 2: Given a differential system having only regular singularities and whose local system formed by its solutions is \mathcal{F} , find a differential system having only regular singularities such that the local system of its solutions is $MC_\lambda(\mathcal{F})$.

As it turns out both problems are closely related to the cohomology of the locally trivial fibration $p_2 : E \rightarrow X$, defined below. The first one is related to the singular cohomology and second one to the de Rham cohomology of p_2 .

Motivated by Katz' description of $MC_\lambda(\mathcal{F})$ in [12], Chap. 5.1, we give a solution to Problem (1) in Section 4. For this, let $X = \mathbb{C} \setminus T$ be as above,

$$E = \{(x, y) \in \mathbb{C}^2 \mid x, y \neq t_i, i = 1, \dots, r, x \neq y\},$$

$p_i : E \rightarrow X$, $i = 1, 2$, be the i -th projection,

$$q : E \rightarrow \mathbb{C}^\times, (x, y) \mapsto y - x,$$

$j : E \rightarrow \mathbb{P}^1(\mathbb{C}) \times X$ the tautological inclusion and $\bar{p}_2 : \mathbb{P}^1(\mathbb{C}) \times X \rightarrow X$ the (second) projection onto X . Moreover, let \mathcal{L}_λ denote the Kummer sheaf associated to the representation, which sends a generator of $\pi_1(\mathbb{C}^\times)$ to λ (see Definition 4.1). The following theorem is proved in Section 4.3, using singular sheaf cohomology (see Theorem 4.4):

Theorem 1.1 *Let \mathcal{F} be a local system on X and $\lambda \in \mathbb{C}^\times \setminus 1$. Then $MC_\lambda(\mathcal{F})$ is isomorphic to the higher direct image sheaf $R^1(\bar{p}_2)_*(j_*(p_1^*(\mathcal{F}) \otimes q^*(\mathcal{L}_\lambda)))$.*

The idea of the proof is to relate the construction of MC_λ to the group (resp. singular) cohomology of the locally trivial fibration $p_2 : E \rightarrow X$, where one can explicitly work with crossed homomorphisms. We do not use the standard base but a twisted base which arises from the use of commutators, also called Pochhammer contours. The Pochhammer contours are crucial in the further investigation of the convolution in terms of Fuchsian systems (see the Theorem 1.2 below and Remark 6.3). Translating Theorem 1.1 into the language of perverse sheaves, one rediscovers Katz' original construction, see [12], 5.1.7.

In Section 6 we consider Problem (2): In [9], Appendix A, an additive version of Katz' functor is defined. It depends on a scalar $\mu \in \mathbb{C}$ and is denoted by mc_μ . By definition, mc_μ is nothing else then a transformation of tuples of matrices

$$(a_1, \dots, a_r) \in (\mathbb{C}^{n \times n})^r \mapsto mc_\mu(a_1, \dots, a_r) \in (\mathbb{C}^{m \times m})^r.$$

Any choice of elements $t_1, \dots, t_r \in \mathbb{C}$, together with a tuple of matrices $\mathbf{a} := (a_1, \dots, a_r) \in (\mathbb{C}^{n \times n})^r$, yields a Fuchsian system

$$D_{\mathbf{a}} : Y' = \sum_{i=1}^r \frac{a_i}{x - t_i} Y.$$

Then, mc_μ translates into a transformation of Fuchsian systems, sending $D_{\mathbf{a}}$ to $D_{mc_\mu(\mathbf{a})}$. This transformation will be called the *middle convolution* of Fuchsian systems. The tuple of monodromy generators of $D_{\mathbf{a}}$ will be denoted by $\text{Mon}(D_{\mathbf{a}})$ (see Section 5.2). One obtains the following result, see Theorem 6.8:

Theorem 1.2 (Riemann-Hilbert correspondence for MC_λ) *Let $\mu \in \mathbb{C} \setminus \mathbb{Z}$, $\lambda = e^{2\pi i \mu}$ and $\mathbf{a} := (a_1, \dots, a_r)$, $a_i \in \mathbb{C}^{n \times n}$, such that $\text{Mon}(D_{\mathbf{a}}) = (A_1, \dots, A_r) \in \text{GL}_n(\mathbb{C})^r$. Assume that*

$$\text{rk}(a_i) = \text{rk}(A_i - 1), \quad \text{rk}(a_1 + \dots + a_r + \mu) = \text{rk}(\lambda \cdot A_1 \cdots A_r - 1)$$

and that $\langle A_1, \dots, A_r \rangle$ generates an irreducible subgroup of $\text{GL}_n(\mathbb{C})$ such that at least two elements A_i are $\neq 1$. Then

$$\text{Mon}(D_{mc_{\mu-1}(\mathbf{a})}) = MC_\lambda(\text{Mon}(D_{\mathbf{a}})).$$

Let $\mathcal{F}_{\mathbf{a}}$ (resp. $\mathcal{F}_{mc_{\mu-1}(\mathbf{a})}$) denote the local system, formed by the solutions of $D_{\mathbf{a}}$ (resp. $D_{mc_{\mu-1}(\mathbf{a})}$). Since $\text{Mon}(D_{\mathbf{a}})$ (resp. $\text{Mon}(D_{mc_{\mu-1}(\mathbf{a})})$) describes the monodromy of the local system $\mathcal{F}_{\mathbf{a}}$ (resp. $\mathcal{F}_{mc_{\mu-1}(\mathbf{a})}$) (see Remark 5.3), Theorem 1.2 yields

$$\mathcal{F}_{mc_{\mu-1}(\mathbf{a})} \cong MC_\lambda(\mathcal{F}_{\mathbf{a}}).$$

Thus we have obtained the Riemann-Hilbert correspondence (Problem 2) under the assumptions of Theorem 1.2. These assumptions are rather mild and can be further weakened (see the remark following Theorem 6.8).

The main idea of the proof of Theorem 1.2 is to use Euler transformations, in order to construct a suitable period matrix I^μ , describing the pairing between the homology and the de Rham cohomology with coefficients in $(\mathfrak{p}_1^*(\mathcal{F}_{\mathbf{a}}) \otimes q^*(\mathcal{L}_\lambda))|_{X(y_0)}$, resp. $(\mathfrak{p}_1^*(\mathcal{F}_{\mathbf{a}}) \otimes q^*(\mathcal{L}_\lambda))^\vee|_{X(y_0)}$, where $X(y_0)$ denotes the fibre of $\mathfrak{p}_2 : E \rightarrow X$ over y_0 . The columns of I^μ are solutions of a Fuchsian system $D_{c_\mu(\mathbf{a})}$ (called the *convolution* of $D_{\mathbf{a}}$ with μ) such that the middle convolution $D_{mc_\mu(\mathbf{a})}$ is a factor system of $D_{c_\mu(\mathbf{a})}$. The rows of I^μ have an interpretation in terms of crossed homomorphisms (already used in Section 4) which makes it possible to compute the monodromy of $D_{c_\mu(\mathbf{a})}$ and $D_{mc_\mu(\mathbf{a})}$.

Finally, we give some applications of our methods (Section 7): From Theorem 1.2, one obtains an algorithm for the construction of Fuchsian systems corresponding to irreducible rigid local systems under the Riemann-Hilbert correspondence, see Section 7.1. As a byproduct, one obtains integral expressions for the solutions of these Fuchsian systems. Compare to the work of Haraoka and Yokoyama ([10], [21]) who use a different approach (in the case of semisimple monodromy) to obtain integral expression of such solutions.

Then we apply Theorem 1.2 to the construction problem of differential systems which arise from geometry: These are differential systems which arise from iterated extensions of subfactors of Gauß-Manin connections (see Section 7.2 for the definition). These differential systems have many favorable properties. For example, under some additional assumptions (the connectivity of motivic Galois groups), such a system satisfies the Grothendieck-Katz p -curvature conjecture which makes it possible to construct the Lie algebra of its differential Galois group from its p -curvatures, see André [3], Theorem 0.7.1.

Using results of André [1], one obtains the following result (Theorem 7.1):

Theorem 1.3 *Let K be a number field, $\mathbf{a} = (a_1, \dots, a_r)$, $a_i \in K^{n \times n}$, $\mu \in \mathbb{Q}$, such that the conditions of Theorem 1.2 hold for $D_{\mathbf{a}}$. If $D_{\mathbf{a}}$ is arising from geometry, then $D_{mc_{\mu}(\mathbf{a})}$ is arising from geometry.*

This makes it possible to construct explicitly a large number of differential systems which arise from geometry. One could start from any differential system with finite monodromy (which automatically arises from geometry) and apply the convolution mc_{μ} , $\mu \in \mathbb{Q}$, to it. In Section 7.2, we consider examples which are derived from Lamé equations with finite monodromy, related to the work of Baldassari [4] and Beukers and van der Waall [5], [18]. This leads to new (non-rigid) examples of differential systems for which the Grothendieck-Katz p -curvature conjecture is true, see Corollary 7.11. See Katz [12], Chap. 9, for a proof of the Grothendieck p -curvature conjecture for Fuchsian systems corresponding to irreducible rigid local systems.

As another application, we investigate the effect of the convolution on the p -curvature (p a prime) of a Fuchsian system defined over \mathbb{Q} . One obtains a simple formula for the computation of the p -curvature matrices (Lemma 7.8) and the following result, see Section 7.3 for definitions and Theorem 7.9:

Theorem 1.4 *Let K be a number field and \mathfrak{p} a prime of K lying over p . Let $\mu \in \mathbb{Q}$ and $\mathbf{a} = (a_1, \dots, a_r)$, $a_i \in K^{n \times n}$, such that the p -curvature matrix $a(\mathfrak{p})$ of $D_{\mathbf{a}}$ satisfies $a(\mathfrak{p})^k = 0$. Then the p -curvature matrix $mc_{\mu}(a(\mathfrak{p}))$ of the convoluted Fuchsian system $D_{mc_{\mu}(\mathbf{a})}$ satisfies $mc_{\mu}(a(\mathfrak{p}))^{k+2} = 0$.*

The crucial observation here is, that the convolution $D_{c_{\mu}(\mathbf{a})}$ is a differential system in Okubo normal form (see Section 7.3 for definition). For these systems there exists a closed formula for the computation of the p -curvature matrices (Lemma 7.8). Theorem 1.4 is interesting in view of the Bombieri-Dwork conjecture which relates the nilpotence of the p -curvatures to the geometric nature of a differential equation, see Section 7.3. Also, information on the p -adic radius of solvability is encoded in the nilpotence degree of the p -curvatures, see [1], Chap. 4.

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2 Definition and properties of the middle convolution functor MC_{λ}

In this section, we recall the algebraic construction of the multiplicative version of the convolution functor defined in [9]. We actually consider a slight modifi-

cation of the multiplicative version of the convolution. This modification is just of formal nature and due to the topological setup used in the later sections.

We will use the following notations and conventions throughout the paper: Let K be a field and G a group. The category of finite dimensional left- G -modules is denoted by $\text{Mod}(K[G])$. Mostly, we do not distinguish notationally between an element of $\text{Mod}(K[G])$ and its underlying vector space. Let V be an element $\text{Mod}(K[G])$ corresponding to a representation $\rho : G \rightarrow \text{GL}(V)$ and W a K vector space such that one has a perfect pairing

$$\langle, \rangle : W \times V \rightarrow K.$$

Then W turns into a G -module, where g acts via the unique linear transformation $\rho^\vee(g)$ such that $\langle \rho^\vee(g)w, \rho(g)v \rangle = \langle w, v \rangle$ for all $w \in W$ and all $v \in V$. We refer to W as the dual module of V with respect to \langle, \rangle and often denote it V^\vee . If for $g \in G$, the linear transformation $\rho(g)$ is a given element $A \in \text{GL}(V)$, then we write $A^\vee \in \text{GL}(W)$ for $\rho^\vee(g)$.

2.1 Definition of MC_λ

Let F_r denote the free group on r generators f_1, \dots, f_r . An element in $\text{Mod}(K[F_r])$ is viewed as a pair (\mathbf{A}, V) , where V is a vector space over K and $\mathbf{A} = (A_1, \dots, A_r)$ is an element of $\text{GL}(V)^r$ such that f_i acts on V via A_i , $i = 1, \dots, r$. For $(\mathbf{A}, V) \in \text{Mod}(K[F_r])$, where $\mathbf{A} = (A_1, \dots, A_r) \in \text{GL}(V)^r$, and $\lambda \in K^\times$ one can construct an element $(C_\lambda(\mathbf{A}), V^r) \in \text{Mod}(K[F_r])$, $C_\lambda(\mathbf{A}) = (B_1, \dots, B_r) \in \text{GL}(V^r)^r$, as follows: For $k = 1, \dots, r$, B_k maps a vector $(v_1, \dots, v_r)^{\text{tr}} \in V^r$ to

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ & \ddots & & \\ & & 1 & \\ \lambda(A_1 - 1) & \dots & \lambda(A_{k-1} - 1) & \lambda A_k & (A_{k+1} - 1) & \dots & (A_r - 1) \\ & & & & 1 & & \\ & & & & & \ddots & \\ 0 & & \dots & & & 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ v_r \end{pmatrix}.$$

We set $C_\lambda(\mathbf{A}) := (B_1, \dots, B_r)$. There are the following $\langle B_1, \dots, B_r \rangle$ -invariant subspaces of V^r :

$$\mathcal{K}_k = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \ker(A_k - 1) \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad (k\text{-th entry}), k = 1, \dots, r,$$

and

$$\mathcal{L} = \cap_{k=1}^r \ker(B_k - 1) = \ker(B_1 \cdots B_r - 1).$$

Let $\mathcal{K} := \oplus_{i=1}^r \mathcal{K}_i$.

If $\lambda \neq 1$, then

$$\mathcal{L} = \left\langle \begin{pmatrix} A_2 \cdots A_r v \\ A_3 \cdots A_r v \\ \vdots \\ v \end{pmatrix} \mid v \in \ker(\lambda \cdot A_1 \cdots A_r - 1) \right\rangle.$$

and

$$\mathcal{K} + \mathcal{L} = \mathcal{K} \oplus \mathcal{L}.$$

Definition 2.1 Let $V = (\mathbf{A}, V) \in \text{Mod}(K[F_r])$.

i) We call the $K[F_r]$ -module $C_\lambda(V) := (C_\lambda(\mathbf{A}), V^r)$ the *convolution* of V with λ .

ii) Let $MC_\lambda(\mathbf{A}) := (\tilde{B}_1, \dots, \tilde{B}_r) \in \text{GL}(V^r / (\mathcal{K} + \mathcal{L}))^r$, where \tilde{B}_k is induced by the action of B_k on $V^r / (\mathcal{K} + \mathcal{L})$. The $K[F_r]$ -module $MC_\lambda(V) := (MC_\lambda(\mathbf{A}), V^r / (\mathcal{K} + \mathcal{L}))$ is called the *middle convolution* of (A_1, \dots, A_r) with λ .

Remark: In [9], we use the same construction, with the difference that the k -th block row of B_k is

$$((A_1 - 1), \dots, (A_{k-1} - 1), \lambda A_k, \lambda(A_{k+1} - 1), \dots, \lambda(A_r - 1)).$$

2.2 Properties of MC_λ

Let $V \rightarrow V'$ be a morphism of F_r -modules. This clearly induces a morphism $C_\lambda(V) \rightarrow C_\lambda(V')$. Since the subspaces \mathcal{K} and \mathcal{L} of $C_\lambda(V)$ are mapped to their corresponding subspaces \mathcal{K}' and \mathcal{L}' of $C_\lambda(V')$ this induces a morphism $MC_\lambda(V) \rightarrow MC_\lambda(V')$. The following proposition is easy to prove, compare to [9], Proposition 2.6 and Lemma 2.8:

Proposition 2.2 Let $\lambda \in K^\times$. The transformation $V \mapsto MC_\lambda(V)$ (resp. $V \mapsto C_\lambda(V)$) is a covariant, end-exact, functor of $\text{Mod}(K[F_r])$ to itself.

Definition 2.3 Let $V = (\mathbf{A}, V) \in \text{Mod}(K[F_r])$, where $\mathbf{A} = (A_1, \dots, A_r) \in \text{GL}(V)^r$. We say that V satisfies (*) if

$$\bigcap_{j \neq i} \ker(A_j - 1) \cap \ker(\tau A_i - 1) = 0, \quad i = 1, \dots, r, \quad \forall \tau \in K^\times.$$

Let $\mathcal{U}_i(\tau) := \sum_{j \neq i} \text{im}(A_j - 1) + \text{im}(\tau A_i - 1)$, $i = 1, \dots, r$, $\tau \in K^\times$. We say that V satisfies (**) if

$$\dim(\mathcal{U}_i(\tau)) = \dim(V), \quad i = 1, \dots, r, \quad \forall \tau \in K^\times.$$

Remark: The conditions (*) and (**) say, that V has no 1-dimensional factors and/or submodules with the property that only one (or none) of the A_i act non-trivially.

Theorem 2.4 Let $V = (\mathbf{A}, V) \in \text{Mod}(K[F_r])$, where $\mathbf{A} = (A_1, \dots, A_r) \in \text{GL}(V)^r$ and $\lambda \in K^\times$.

i) If $\lambda \neq 1$, then

$$\dim(MC_\lambda(V)) = \sum_{k=1}^r \text{rk}(A_k - 1) - (\dim(V) - \text{rk}(\lambda \cdot A_1 \dots A_r - 1)).$$

ii) If $\lambda_1, \lambda_2 \in K^\times$ such that $\lambda_1 \lambda_2 = \lambda$ and (*) and (**) hold for V , then

$$MC_{\lambda_2} MC_{\lambda_1}(V) \cong MC_\lambda(V).$$

iii) Under the assumptions of ii), if V is irreducible, then $MC_\lambda(V)$ is irreducible.

iv) Let $\mathcal{B}_r = \langle Q_1, \dots, Q_{r-1} \rangle$ be the abstract Artin braid group, where the generators Q_1, \dots, Q_{r-1} of \mathcal{B}_r act in the following way on tuples $(g_1, \dots, g_r) \in G^r$ (where G is a group):

$$(1) \quad Q_i(g_1, \dots, g_r) = (g_1, \dots, g_{i-1}, g_i g_{i+1} g_i^{-1}, g_i, g_{i+2}, \dots, g_r), \quad i = 1, \dots, r-1.$$

For any $Q \in \mathcal{B}_r$ there exists a $B \in \text{GL}(V^r / (\mathcal{K} + \mathcal{L}))$ such that

$$MC_\lambda(Q(\mathbf{A})) = Q(MC_\lambda(\mathbf{A}))^B,$$

where B acts via component-wise conjugation.

v) Let $K = \mathbb{C}$, $\lambda \in \mathbb{C}$ be a root of unity and $MC_\lambda(\mathbf{A}) = (\tilde{B}_1, \dots, \tilde{B}_r)$. If $\langle A_1, \dots, A_r \rangle$ respects an hermitean form, then $\langle \tilde{B}_1, \dots, \tilde{B}_r \rangle$ respects an hermitean form.

vi) Let the characteristic of K be different from 2 and $MC_{-1}(\mathbf{A}) = (\tilde{B}_1, \dots, \tilde{B}_r)$. If $\langle A_1, \dots, A_r \rangle$ respects an orthogonal (resp. symplectic) form, then $\langle \tilde{B}_1, \dots, \tilde{B}_r \rangle$ respects a symplectic (resp. orthogonal) form.

Proof: i)-iv) follow analogously to [9], Lemma 2.7, Lemma A.4, Theorem 3.5, Corollary 3.6 and Theorem 5.1 (in this order). The claims v) and vi) follow from Lemma 2.5 below. \square

Remark: The Jordan canonical forms of \tilde{B}_k can be computed as in [12], Chap. 6 (using [9], Lemma 4.1).

Lemma 2.5 *Let $\mathbf{A} = (A_1, \dots, A_r)$, $A_k \in \mathrm{GL}_n(K)$, $\lambda \in K^\times$ and $C_\lambda(\mathbf{A}) = (B_1, \dots, B_k)$. Let \mathfrak{G} be an invariant form under A_i , i.e. $A_i^{\mathrm{tr}} \mathfrak{G} A_i = \mathfrak{G}$, $i = 1, \dots, r$. Then*

$$B_k^{\mathrm{tr}} \mathfrak{H} B_k = \mathfrak{H}, \quad k = 1, \dots, r,$$

where

$$\mathfrak{H}_{i,i} = \mathfrak{G} \lambda^{1/2} (A_i^{-1} - 1) (A_i - \lambda^{-1})$$

and

$$\begin{aligned} \mathfrak{H}_{i,j} &= \mathfrak{G} \lambda^{-1/2} (A_i^{-1} - 1) (A_j - 1), \quad \text{if } i < j, \\ \mathfrak{H}_{i,j} &= \mathfrak{G} \lambda^{1/2} (A_i^{-1} - 1) (A_j - 1), \quad \text{if } i > j. \end{aligned}$$

3 The underlying fibration and its cohomology

We fix a finite set $T := \{t_1, \dots, t_r\} \subseteq \mathbb{C}$ such that $t_i \neq t_j$ for $i \neq j$, and set $X := \mathbb{C} \setminus T$. Let W be a topological space and $I := [0, 1]$. A path in W is a continuous map $\gamma : I \rightarrow W$. If γ_1, γ_2 are paths in W such that the endpoint of γ_2 coincides with the initial point of γ_1 , then their product is denoted by $\gamma_1 \gamma_2$. If γ is a closed path in W with initial point w_0 , then $\gamma \in \pi_1(W, w_0)$ will also denote the corresponding homotopy class.

3.1 The underlying fibration

In this subsection we study a fibration whose cohomology will lead to the geometric interpretation of C_λ and MC_λ in Subsection 4.3. The contents of this section are well known, compare to [6], Chap. 1, and [19].

For $n \in \mathbb{N}$, consider the configuration space

$$\mathcal{O}_n := \{P \subseteq \mathbb{C} \mid |P| = n\}$$

of subsets of \mathbb{C} of cardinality equal to n . Let further

$$\mathcal{O}^n := \{(p_1, \dots, p_n) \in \mathbb{C}^n \mid i \neq j \Rightarrow p_i \neq p_j\}.$$

Since the map

$$\mathcal{O}^n \rightarrow \mathcal{O}_n, (p_1, \dots, p_n) \mapsto \{p_1, \dots, p_n\},$$

is an unramified covering map (where \mathcal{O}_n is equipped with the obvious topology), we will consider $\mathcal{B}^n := \pi_1(\mathcal{O}^n, (b_1, \dots, b_n))$ as a subgroup of $\mathcal{B}_n := \pi_1(\mathcal{O}_n, \{b_1, \dots, b_n\})$ via covering theory.

It is well known that the fundamental group \mathcal{B}_n is isomorphic to the abstract Artin braid group, i.e., it has a presentation on $n - 1$ generators Q_1, \dots, Q_{n-1} subject to the *braid relations*

$$\begin{aligned} Q_i Q_j &= Q_j Q_i && \text{if } |i - j| > 1, \\ Q_i Q_{i+1} Q_i &= Q_{i+1} Q_i Q_{i+1} && \text{for } i = 1, \dots, n - 2. \end{aligned}$$

The group \mathcal{B}^n is isomorphic to the (abstract) pure Artin braid group and generated by the elements

$$Q_{i,j} := (Q_i^2)^{Q_{i+1}^{-1} \cdots Q_{j-1}^{-1}} = (Q_{j-1}^2)^{Q_{j-2} \cdots Q_i},$$

where $1 \leq i < j \leq n$.

Let now $n = r + 2$. We set $\gamma_i := Q_{1,i+1}$, $i = 1, \dots, r + 1$, and $\delta_k := Q_{k+1,r+2}$, $k = 1, \dots, r$. Using the braid relations, one can easily see that for $k = 1, \dots, r$, the following formula holds:

$$(\delta_k \gamma_1, \dots, \delta_k \gamma_{r+1}) = (\gamma_1, \dots, \gamma_{k-1}, \gamma_k^{\gamma_{r+1}}, \gamma_{k+1}^{[\gamma_k, \gamma_{r+1}]}, \dots, \gamma_r^{[\gamma_k, \gamma_{r+1}]}, \gamma_{r+1}^{\gamma_k \gamma_{r+1}}),$$

where $[\gamma_k, \gamma_{r+1}] = \gamma_k^{-1} \gamma_{r+1}^{-1} \gamma_k \gamma_{r+1}$ and $\delta_k \gamma_1 = \delta_k \gamma \delta_k^{-1}$, see also [6], 1.8.3, and [19].

Let $T = \{t_1, \dots, t_r\} \subseteq \mathbb{C}$, $X := \mathbb{C} \setminus T$ and

$$E := \{(x, y) \in \mathbb{C}^2 \mid x, y \neq t_i, i = 1, \dots, r, x \neq y\}.$$

The second projection $p_2 : E \rightarrow X$ is a locally trivial fibration. The fibre over y is denoted by $X(y_0)$ and is via the first projection identified with $X \setminus \{y_0\}$. One has a commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{p_2} & X \\ \downarrow & & \downarrow \\ \mathcal{O}^{r+2} & \xrightarrow{p} & \mathcal{O}^{r+1} \end{array},$$

where $p(p_1, \dots, p_{r+2}) := (p_2, \dots, p_{r+2})$ and the first (resp. second) vertical arrow is given by $(x, y) \mapsto (x, t_1, \dots, t_r, y)$ (resp. $y \mapsto (t_1, \dots, t_r, y)$).

The long exact sequences of homotopy groups, associated to locally trivial fibrations, lead then to a commutative diagram

$$\begin{array}{ccccccc} 1 & \rightarrow & \pi_1(X(y_0), x_0) & \rightarrow & \pi_1(E, (x_0, y_0)) & \rightarrow & \pi_1(X, y_0) \rightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \rightarrow & \pi_1(\mathcal{F}_r, (x_0, t_1, \dots, y_0)) & \rightarrow & \mathcal{B}^{r+2} & \rightarrow & \mathcal{B}^{r+1} \rightarrow 1 \end{array},$$

where $\mathcal{B}^{r+2} = \pi_1(\mathcal{O}^{r+2}, (x_0, t_1, \dots, t_r, y_0))$, $\mathcal{B}^{r+1} = \pi_1(\mathcal{O}^{r+1}, (t_1, \dots, t_r, y_0))$ and \mathcal{F}_r denotes the the fibre over (t_1, \dots, t_r, y_0) . It is well known, that the rows are split exact sequences and the vertical arrows are injective, see [6]. Moreover,

one can check that $\pi_1(\mathcal{F}_r, (x_0, t_1, \dots, t_r, y_0))$ is generated by $\gamma_1, \dots, \gamma_{r+1}$ and that the image of $\pi_1(X, y_0)$ in \mathcal{B}^{r+1} is generated by $\delta_1, \dots, \delta_r$.

We define $\alpha_1, \dots, \alpha_{r+1} \in \pi_1(X(y_0), x_0)$ (resp. $\beta_1, \dots, \beta_r \in \pi_1(X, y_0)$) to be the inverse images of $\gamma_1, \dots, \gamma_{r+1}$ (resp. $\delta_1, \dots, \delta_r$) under the first (resp. third) vertical arrow. Thus one deduces that for $k = 1, \dots, r$ the following formula holds:

$$(\beta_k \alpha_1, \dots, \beta_k \alpha_{r+1}) = (\alpha_1, \dots, \alpha_{k-1}, \alpha_k^{\alpha_{r+1}}, \alpha_{k+1}^{[\alpha_k, \alpha_{r+1}]}, \dots, \alpha_r^{[\alpha_k, \alpha_{r+1}]}, \alpha_{r+1}^{\alpha_k \alpha_{r+1}}).$$

3.2 Group cohomology of the fibration

If G is a group and $\rho \rightarrow \text{GL}(V)$ is a representation, then we define the *cohomology of G with values in the module V* to be $H^1(G, V) := C^1(G, V)/B^1(G, V)$, where

$$C^1(G, V) := \{(\delta : G \rightarrow V) \mid \delta(gg') = \delta(g') + \rho(g')^{-1}\delta(g), \forall g, g' \in G\}$$

is the vector space of *crossed homomorphisms* and

$$B^1(G, V) = \{(\delta : G \rightarrow V) \mid \exists v \in V, \delta(g) = v - \rho(g)^{-1}v, \forall g \in G\}$$

is the subspace of *exact crossed homomorphisms*.

Let V be a $\pi_1(X, x_0)$ -module, where α_i acts via $A_i \in \text{GL}(V)$, $i = 1, \dots, r$. Let $\lambda \in \mathbb{C}$, $\Pi := \pi_1(X(y_0), x_0) = \langle \alpha_1, \dots, \alpha_{r+1} \rangle$, and V_λ be the Π -module, whose underlying vectorspace is V and where $\alpha_1, \dots, \alpha_r$ act via $A_i \in \text{GL}(V)$ and α_{r+1} acts via λ . The underlying representation is denoted by ρ_λ .

Definition 3.1 The linear map

$$\tau : C^1(\Pi, V_\lambda) \rightarrow (V_\lambda)^r, \delta \mapsto (\delta([\alpha_{r+1}, \alpha_1]), \dots, \delta([\alpha_{r+1}, \alpha_r]))^{\text{tr}}$$

is called the *twisted evaluation map*.

Lemma 3.2 *If $\lambda \neq 1$, then the kernel of the twisted evaluation map $\tau : C^1(\Pi, V_\lambda) \rightarrow V_\lambda^r$ is $B^1(\Pi, V_\lambda)$.*

Proof: The crossed homomorphism relation implies

$$\delta([\alpha_{r+1}, \alpha_i]) = (1 - \lambda^{-1})\delta(\alpha_i) - (1 - A_i^{-1})\delta(\alpha_{r+1}).$$

So, if $\delta([\alpha_{r+1}, \alpha_i]) = 0$ for $i = 1, \dots, r$, then

$$\delta(\alpha_i) = \frac{1}{1 - \lambda^{-1}}(1 - A_i^{-1})\delta(\alpha_{r+1}), \quad i = 1, \dots, r.$$

An easy induction shows that

$$\delta(\gamma) = \frac{1}{1 - \lambda^{-1}}(1 - \rho_\lambda(\gamma)^{-1})\delta(\alpha_{r+1}),$$

so δ is exact. On the other hand, any vector in V_λ occurs as $\delta(\alpha_{r+1})$ for some $\delta \in C^1(\Pi, V_\lambda)$. Therefore, the claim follows from dimension reasons. \square

Since

$$(\beta_k \alpha_1, \dots, \beta_k \alpha_{r+1}) = (\alpha_1, \dots, \alpha_{k-1}, \alpha_k^{\alpha_{r+1}}, \alpha_{k+1}^{[\alpha_k, \alpha_{r+1}]}, \dots, \alpha_r^{[\alpha_k, \alpha_{r+1}]}, \alpha_{r+1}^{\alpha_k \alpha_{r+1}})$$

and $\rho_\lambda(\alpha_{r+1}) = \lambda$, the map which sends δ to $\delta \circ \beta_k^{-1}$ is contained in $\text{GL}(C^1(\Pi, V_\lambda))$. Thus, by Lemma 3.2, the association

$$\beta[\delta] := [\delta \circ \beta^{-1}]$$

imposes the structure of a $\pi_1(X, y_0)$ -module on $H^1(\Pi, V_\lambda)$ and, by the same arguments, on $H^1(\Pi, V_\lambda^\vee)$.

Consider the pairing

$$(V_\lambda^\vee)^r \times V_\lambda^r \rightarrow \mathbb{C}, ((w_1, \dots, w_r), (v_1, \dots, v_r)^{\text{tr}}) \mapsto \langle w_1, v_1 \rangle + \dots + \langle w_r, v_r \rangle.$$

Let $C_\lambda(\mathbf{A}) = (B_1, \dots, B_r)$ and $\tilde{C}_\lambda(V_\lambda)$ the $\pi_1(X, y_0)$ -module whose underlying vector space is V_λ^r , on which β_k acts via B_k . Let further $\tilde{C}_\lambda(V_\lambda)^\vee$ denote the dual module with respect to the above pairing.

Theorem 3.3 *The linear map*

$$H^1(\Pi, V_\lambda^\vee) \rightarrow \tilde{C}_\lambda(V_\lambda)^\vee, [\delta] \mapsto \tau(\delta)$$

is an isomorphism of $\pi_1(X, y_0)$ -modules.

Proof: It suffices to show that

$$\begin{aligned} \tau(\beta_k^{-1}(\delta)) &= (\delta([\beta_k \alpha_{r+1}, \beta_k \alpha_1], \dots, \delta([\beta_k \alpha_{r+1}, \beta_k \alpha_r]))) \\ (2) \quad &= (\delta([\alpha_{r+1}, \alpha_1], \dots, \delta([\alpha_{r+1}, \alpha_r])) \circ B_k \\ &= \beta_k^{-1}(\tau(\delta)), \end{aligned}$$

for all $\delta \in C^1(\Pi, V_\lambda^\vee)$ and $k = 1, \dots, r$; where the first and the last equality hold by definition. Equality (2) follows from an elementary but tedious computation, using the crossed homomorphism relation and the action of β_k on $(\alpha_1, \dots, \alpha_r)$. \square

4 Convolution of local systems

It is the aim of this section to give an interpretation of the multiplicative version of the convolution in terms of the cohomology of local systems on the punctured sphere.

4.1 Local systems

Let W be a connected topological manifold. A (*complex*) *local system of rank n* on W is a sheaf \mathcal{F} of complex vector spaces which is locally isomorphic to the constant sheaf \mathbb{C}^n . The category of local systems on W is denoted by $\text{LocSys}(W)$. It is closed under tensor product and taking duals. The dual local system of a local system \mathcal{F} on W is denoted by \mathcal{F}^\vee . The stalk of \mathcal{F} at $w_0 \in W$ is denoted by \mathcal{F}_{w_0} .

If γ is a closed path in W starting at w_0 , then there exists a unique linear transformation $\text{Mon}(\gamma)$ such that the stalk $\gamma^*(\mathcal{F})_1$ is canonically isomorphic to $\text{Mon}(\gamma) \cdot \gamma^*(\mathcal{F})_0$. Composition of paths gives rise to the monodromy representation (see [8]):

$$\text{Mon} := \text{Mon}(\mathcal{F}) : \pi_1(W, w_0) \rightarrow \text{GL}(\mathcal{F}_{w_0}).$$

It is well known, that the functor

$$\text{LocSys}(W) \rightarrow \text{Mod}(\mathbb{C}[\pi_1(W, w_0)]), \mathcal{F} \mapsto \mathcal{F}_{w_0}$$

is an equivalence of categories, see [8], Corollaire 1.4.

4.2 The middle convolution functor MC_λ for local systems

Definition 4.1 Let γ be a closed path in \mathbb{C}^\times which has initial point x_0 and encircles 0 once in counterclockwise direction, $\lambda \in \mathbb{C}^\times$ and

$$\chi : \pi_1(\mathbb{C}^\times, x_0) \rightarrow \text{GL}(\mathbb{C}), \gamma \mapsto \lambda.$$

The local system on \mathbb{C}^\times corresponding to the (module associated to the) homomorphism χ is called the *Kummer sheaf* associated to λ and y_0 and is denoted by \mathcal{L}_λ .

Let $\alpha_1, \dots, \alpha_{r+1}$ (resp. β_1, \dots, β_r) be as in the previous subsections and \mathcal{F} be the local system associated to the representation

$$\rho : \pi_1(X, x_0) \rightarrow \text{GL}(V), \alpha_i \mapsto A_i, i = 1, \dots, r.$$

For $\lambda \in \mathbb{C}$, let $C_\lambda(\mathbf{A}) = (B_1, \dots, B_r) \in \text{GL}(V^r)^r$ and $MC_\lambda(\mathbf{A}) = (\tilde{B}_1, \dots, \tilde{B}_r) \in \text{GL}(V^r/(\mathcal{K} + \mathcal{L}))^r$. We define $C_\lambda(\mathcal{F})$ to be the local system associated to

$$\pi_1(X, x_0) \rightarrow \text{GL}(V^r), \beta_i \mapsto B_i, i = 1, \dots, r.$$

Similarly, let $MC_\lambda(\mathcal{F})$ be the local system associated to

$$\pi_1(X, x_0) \rightarrow \text{GL}(V^r/(\mathcal{K} + \mathcal{L})), \beta_i \mapsto \tilde{B}_i, i = 1, \dots, r.$$

Proposition 4.2 *The local system $C_\lambda(\mathcal{F})$, resp. $MC_\lambda(\mathcal{F})$, is canonically isomorphic to the local system corresponding to the representation*

$$\pi_1(X, x_0) \rightarrow \text{GL}(V), \alpha_i \mapsto B_i, i = 1, \dots, r,$$

resp.

$$\pi_1(X, x_0) \rightarrow \text{GL}(V), \alpha_i \mapsto \tilde{B}_i, i = 1, \dots, r.$$

Proof: There exists, up to homotopy, a unique path γ in X with initial point y_0 and endpoint x_0 such that $\beta_i = \gamma^{-1}\alpha_i\gamma$ for $i = 1, \dots, r$. This path induces canonical isomorphisms

$$C_\lambda(\mathcal{F})|_{y_0} \rightarrow C_\lambda(\mathcal{F})|_{x_0},$$

resp.

$$MC_\lambda(\mathcal{F})|_{y_0} \rightarrow MC_\lambda(\mathcal{F})|_{x_0},$$

which are compatible with the induced isomorphism of fundamental groups

$$\pi_1(X, y_0) \rightarrow \pi_1(X, x_0), \beta_i \mapsto \alpha_i = \gamma\alpha_i\gamma^{-1}$$

and the action of the two fundamental groups on their respective fibers. \square

In view of Proposition 2.2 one obtains covariant, end-exact, functors

$$C_\lambda : \text{LocSys}(X) \rightarrow \text{LocSys}(X), \mathcal{F} \mapsto C_\lambda(\mathcal{F})$$

and

$$MC_\lambda : \text{LocSys}(X) \rightarrow \text{LocSys}(X), \mathcal{F} \mapsto MC_\lambda(\mathcal{F}).$$

Moreover, all the properties of MC_λ , as given in Theorem 2.4, immediately translate into the language of local systems.

The following definition is justified by the results of the next subsection:

Definition 4.3 The local system $C_\lambda(\mathcal{F})$ (resp. $MC_\lambda(\mathcal{F})$) is called the *convolution* (resp. *middle convolution*) of \mathcal{F} with \mathcal{L}_λ .

4.3 Cohomological interpretation of MC_λ

Let \mathcal{S} a sheaf of complex vector spaces on W . An i -cochain ψ is a map which associates to any i -chain $\sigma : \Delta^i \rightarrow W$ an element $\psi(\sigma) \in \mathbb{C}$. The set of i -cochains is denoted by $C^i(W)$. Consider the (injective and torsionless, see [20]) resolution of the constant sheaf \mathbb{C} on W via cochains

$$0 \rightarrow \mathbb{C} \rightarrow C^0(W) \rightarrow C^1(W) \rightarrow C^2(W) \rightarrow \dots$$

and let $H^i(W, \mathcal{S}) := H^i(\Gamma(C^* \otimes \mathcal{S}))$.

Let $p : W_1 \rightarrow W_2$ be a continuous map of topological spaces and \mathcal{S} a sheaf on W_1 . The sheaf associated to the presheaf

$$U \mapsto S_U := H^i(p^{-1}(U), \mathcal{S}|_U) \quad (U \text{ open in } W_2)$$

is denoted by $R^i p_*(\mathcal{S})$ (it is well known that $R^i p_*$ can be viewed as the i -th higher direct image functor of p_*).

Let

$$E = \{(x, y) \in \mathbb{C}^2 \mid x, y \neq t_i, i = 1, \dots, r, x \neq y\},$$

$p_i : E \rightarrow X$, $i = 1, 2$, be the i -th projection,

$$q : E \rightarrow \mathbb{C}^\times, (x, y) \mapsto y - x,$$

$j : E \rightarrow \mathbb{P}^1(\mathbb{C}) \times X$ the tautological inclusion and $\bar{p}_2 : \mathbb{P}^1(\mathbb{C}) \times X \rightarrow X$ the (second) projection onto X .

Theorem 4.4 *Let \mathcal{F} be a local system on X , $\lambda \in \mathbb{C}^\times \setminus 1$ and \mathcal{L}_λ the Kummer sheaf associated to λ . Then*

$$MC_\lambda(\mathcal{F}) \cong R^1(\bar{p}_2)_*(j_*(p_1^*(\mathcal{F}) \otimes q^*(\mathcal{L}_\lambda))).$$

Proof: Let \mathcal{F} be the local system associated to a representation

$$\pi_1(X, x_0) \rightarrow \mathrm{GL}(V), \alpha_i \mapsto A_i, i = 1, \dots, r,$$

where α_i is as in Section 4.1. Let $\mathcal{G} := p_1^*(\mathcal{F}) \otimes q^*(\mathcal{L}_\lambda)$ and \mathcal{G}_{y_0} the restriction of \mathcal{G} to $X(y_0)$ (thus \mathcal{G}_{y_0} corresponds to the Π -module V_λ of the last section). Let $\psi \in C^1(X(y_0), \mathcal{G}_{y_0}^\vee)$ be a closed cochain and σ_1, σ_2 closed paths in $X(y_0)$, based at x_0 . By definition,

$$\psi(\sigma_1\sigma_2)|_0 = \psi(\sigma_2)|_0 + \mathrm{Mon}(\sigma_2)^{-1}\psi(\sigma_1)|_0.$$

This induces an isomorphism

$$H : H^1(X(y_0), \mathcal{G}_{y_0}^\vee) \rightarrow H^1(\Pi, V_\lambda^\vee), [\psi] \mapsto [(\sigma \mapsto \psi(\sigma)|_0)].$$

Since $p_2 : E \rightarrow X$ is a locally trivial fibration, $R^1(p_2)_*(\mathcal{G}^\vee)$ is a local system. By construction, the monodromy action of $\beta_k \in \pi_1(X, y_0)$ on $R^1(p_2)_*(\mathcal{G}^\vee)|_{y_0} = H^1(X(y_0), \mathcal{G}_{y_0}^\vee)$ is the one which is induced by sending α_i to $\beta_k^{-1}\alpha_i$. This yields a canonical isomorphism of $\pi_1(X, y_0)$ -modules between $H^1(X(y_0), \mathcal{G}_{y_0}^\vee)$ and $H^1(\Pi, V_\lambda^\vee)$. Thus, by Theorem 3.3, one has a canonical isomorphism

$$(3) \quad R^1(p_2)_*(\mathcal{G}^\vee) \cong C_\lambda(\mathcal{F})^\vee.$$

Consider the following subspaces of $(V_\lambda^\vee)^r = C_\lambda(\mathcal{F})^\vee|_{y_0}$:

$$V_1 := (\mathrm{im}((A_1^\vee)^{-1} - 1), \dots, \mathrm{im}((A_r^\vee)^{-1} - 1))$$

and

$$V_2 := \{(w_1, w_2, \dots, w_r) \in (V_\lambda^\vee)^r \mid \sum_{i=1}^{r-1} ((A_{i+1}^\vee \cdots A_r^\vee)^\vee)^{-1} w_i + w_r \in \mathrm{im}((A_1^\vee \cdots A_r^\vee)^{-1} \lambda - 1)\}.$$

One can easily check that $V_1 \cap V_2 \leq (V_\lambda^\vee)^r$ is the $\pi_1(X, y_0)$ -submodule which corresponds to $MC_\lambda(\mathcal{F})^\vee$.

The image of the cohomology with compact supports $H_c^1(X(y_0), \mathcal{G}_{y_0}^\vee)$ in $H^1(X(y_0), \mathcal{G}_{y_0}^\vee)$ is mapped under $\tau \circ H$ isomorphically onto $V_1 \cap V_2$. This can be seen using similar arguments as Shimura [17], Chap. 8, or by writing α_i as the product $\bar{\gamma}_i^{-1} \hat{\gamma}_i \bar{\gamma}_i$ (where $\bar{\gamma}_i$ is a path which starts at x_0 and goes near to the singularity t_i , and $\hat{\gamma}_i$ moves along a small circle around t_i) and using the compact supports condition at t_1, \dots, t_k and ∞ . The image of $H_c^1(X(y_0), \mathcal{G}_{y_0}^\vee)$ in $H^1(X(y_0), \mathcal{G}_{y_0}^\vee)$ is canonically isomorphic to $H^1(j_*(\mathcal{G}_{y_0}^\vee))$ (see [15], Lemma 5.3). Therefore,

$$MC_\lambda(\mathcal{F})^\vee \cong R^1(\bar{p}_2)_*(j_*(\mathcal{G}^\vee)).$$

Finally, the Poincaré pairing yields an isomorphism

$$R^1(\bar{p}_2)_*(j_*(\mathcal{G}^\vee))^\vee \cong R^1(\bar{p}_2)_*(j_*(\mathcal{G})) = R^1(\bar{p}_2)_*(j_*(p_1^*(\mathcal{F}) \otimes q^*(\mathcal{L}_\lambda)))$$

(see e.g. [15], Lemma 5.3). \square

Remark 4.5 i) The resolution via singular cochains allows one to use ground fields different from \mathbb{C} as coefficients of cohomology. One could even work in the category of local systems over principal ideal domains, see [20].

5 The middle convolution transformation mc_μ of Fuchsian systems

5.1 Definition of mc_μ for tuples of matrices

In this section we recall the additive convolution as given in [9], App. A.

Let K be any field and $\mathbf{a} = (a_1, \dots, a_r)$, $a_k \in K^{n \times n}$. For $\mu \in K$ one can define blockmatrices b_k , $k = 1, \dots, r$, as follows:

$$b_k := \begin{pmatrix} 0 & & & \dots & & & 0 \\ & \ddots & & & & & \\ a_1 & \dots & a_{k-1} & a_k + \mu & a_{k+1} & \dots & a_r \\ & & & & \ddots & & \\ 0 & & & \dots & & & 0 \end{pmatrix} \in K^{nr \times nr},$$

where b_k is zero outside the k -th block row.

There are the following left- $\langle b_1, \dots, b_r \rangle$ -invariant subspaces of the column vector space K^{nr} (with the tautological action of $\langle b_1, \dots, b_r \rangle$):

$$\mathfrak{k}_k = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \ker(a_k) \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad (k\text{-th entry}), k = 1, \dots, r,$$

and

$$\mathfrak{l} = \bigcap_{k=1}^r \ker(b_k) = \ker(b_1 + \dots + b_r).$$

Let $\mathfrak{k} := \bigoplus_{k=1}^r \mathfrak{k}_k$.

If $\mu \neq 0$ then

$$\mathfrak{l} = \left\langle \begin{pmatrix} v \\ \vdots \\ v \end{pmatrix} \mid v \in \ker(a_1 + \dots + a_r + \mu) \right\rangle.$$

and

$$\mathfrak{k} + \mathfrak{l} = \mathfrak{k} \oplus \mathfrak{l}.$$

We fix an isomorphism I between $K^{nr}/(\mathfrak{k} + \mathfrak{l})$ and K^m .

Definition 5.1 We call $c_\mu(\mathbf{a}) := (b_1, \dots, b_r)$ the (*additive version of the*) convolution of $\mathbf{a} = (a_1, \dots, a_r)$ with μ . The tuple of matrices $mc_\mu := (\tilde{b}_1, \dots, \tilde{b}_r) \in K^{m \times m}$, where \tilde{b}_i is induced by the action of b_i on $K^m (\simeq K^{nr}/(\mathfrak{k} + \mathfrak{l}))$, is called the (*additive version of the*) middle convolution of \mathbf{a} with μ .

5.2 The definition of mc_μ for Fuchsian systems and monodromy of differential systems

Let $T := \{t_1, \dots, t_r\}$, $X := \mathbb{C} \setminus T$ and $\mathbf{c} := (c_1, \dots, c_r)$, $c_i \in \mathbb{C}^{k \times k}$. The Fuchsian system

$$Y' = \sum_{i=1}^r \frac{c_i}{x - t_i} Y$$

is denoted by $D_{\mathbf{c}}$.

Definition 5.2 Let $\mathbf{a} := (a_1, \dots, a_r)$, $a_i \in \mathbb{C}^{n \times n}$, and $\mu \in \mathbb{C}$. The Fuchsian system $D_{c_\mu(\mathbf{a})}$ (resp. $D_{mc_\mu(\mathbf{a})}$) is called the convolution (resp. middle convolution) of $D_{\mathbf{a}}$ with μ .

Let $\gamma_1, \dots, \gamma_{r+1}$ be a homotopy base of $\pi_1(X, o)$, D a linear system of differential equations which has no singularities in X and F a fundamental system of D , consisting of functions which are defined in a small neighborhood of o .

Analytic continuation of F along γ_i transforms F into $F \cdot \text{Mon}(\gamma_i)$. We call the tuple

$$\text{Mon}(D) := (\text{Mon}(\gamma_1), \dots, \text{Mon}(\gamma_r))$$

the tuple of *monodromy generators* of D with respect to F and $\gamma_1, \dots, \gamma_r$.

Remark 5.3 i) An element $\gamma_i \in \pi_1(X, o)$ acts (via $\text{Mon}(\gamma_i)$) from the right on the vector space \mathcal{S} spanned by the rows of the fundamental system F of D . Let \mathcal{F} denote the local system \mathcal{F} formed by the solutions of D (locally at o given by the columns of F) and fix the isomorphism

$$\mathcal{F}_o \rightarrow \mathbb{C}^n, f_i(o) \mapsto e_k,$$

where f_i denotes the i -th column of F and e_k is the k -th standard vector of \mathbb{C}^n . Then the monodromy of \mathcal{F} with respect to γ is given by the same matrix $\text{Mon}(\gamma_i)$ acting from the left on $\mathcal{F}_0 \simeq \mathbb{C}^n$.

ii) As a factor system of $D_{c_\mu(\mathbf{a})}$, the middle convolution $D_{mc_\mu(\mathbf{a})}$, $mc_\mu(\mathbf{a}) \in (\mathbb{C}^{m \times m})^r$, can be constructed by a base change, transforming a basis of $\mathfrak{k} + \mathfrak{l}$ to the first $nr - m$ standard vectors, and cutting out the $m \times m$ -block matrices corresponding to the last m entries. The same construction applies for a fundamental matrix of $D_{c_\mu(\mathbf{a})}$. We say that a matrix whose columns are solutions of $D_{c_\mu(\mathbf{a})}$ (not necessarily a fundamental matrix of $D_{c_\mu(\mathbf{a})}$) gives rise to a *fundamental matrix of $D_{mc_\mu(\mathbf{a})}$* , if the resulting matrix under the above construction of transforming and cutting out is a fundamental matrix of $D_{mc_\mu(\mathbf{a})}$.

6 Compatibility of MC_λ and mc_μ

In this section we relate the additive version of the convolution to the multiplicative version (Subsection 6.2).

6.1 The Euler transform

A commutator

$$[\alpha_i, \alpha_j] = \alpha_i^{-1} \alpha_j^{-1} \alpha_i \alpha_j$$

is called a *Pochhammer contour*. Pochhammer contours are widely used in the theory of ordinary differential equations, see [11], [14] and [22].

Definition 6.1 Let $\mu \in \mathbb{C}$, $g := (g_{i,j})$ be a matrix whose entries $g_{i,j}$ are (multi-valued) functions which are holomorphic on X . The path α_{r+1} encircles an open neighbourhood U of y_0 . The matrix valued function

$$I_{[\alpha_{r+1}, \alpha_i]}^\mu(g)(y) := \int_{[\alpha_{r+1}, \alpha_i]} g(x)(y-x)^{\mu-1} dx, \quad y \in U,$$

is called the *Euler transform* of g with respect to $[\alpha_{r+1}, \alpha_i]$ and μ .

The next lemma shows that the Euler transformation is compatible with the convolution:

Lemma 6.2 *Let $\mathbf{a} := (a_1, \dots, a_r)$, $a_i \in \mathbb{C}^{n \times n}$, and $\mu_1, \mu_2 \in \mathbb{C}$. If $g(x)$ is a solution of $D_{c_{\mu_1}(\mathbf{a})}$, then $I_{[\alpha_{r+1}, \alpha_i]}^{\mu_2}(g)(y)$ is a solution for $D_{c_{\mu_1+\mu_2}(\mathbf{a})}$, where y is contained in an open neighborhood of y_0 which is encircled by α_{r+1} .*

Proof: In the following, we omit the subscript $[\alpha_{r+1}, \alpha_i]$ at the integral sign. For $y \in U$,

$$\begin{aligned}
(y-T) \frac{d(I_{[\alpha_{r+1}, \alpha_i]}^{\mu_2}(g))}{dy} &= (y-T) \int \frac{d}{dy} g(x)(y-x)^{\mu_2-1} dx \\
&= \int ((y-x) + (x-T)) \left(\frac{d}{dy} g(x)(y-x)^{\mu_2-1} \right) dx \\
(4) \qquad &= (\mu_2-1) I_{[\alpha_{r+1}, \alpha_i]}^{\mu_2}(g)(y) \\
&\quad + (\mu_2-1) \int (x-T) g(x)(y-x)^{\mu_2-2} dx,
\end{aligned}$$

where one is allowed to differentiate under the integration sign since $[\alpha_{r+1}, \alpha_i]$ is compact. One has

$$\begin{aligned}
(5) \qquad 0 &= \int \frac{d}{dx} ((x-T)g(x)(y-x)^{\mu_2-1}) dx \\
&= \int g(x)(y-x)^{\mu_2-1} dx + \\
&\quad \int (x-T)g'(x)(y-x)^{\mu_2-1} dx - \\
&\quad (\mu_2-1) \int (x-T)g(x)(y-x)^{\mu_2-2} dx.
\end{aligned}$$

Therefore

$$\begin{aligned}
(\mu_2-1) \int (x-T)g(x)(y-x)^{\mu_2-2} dx &= \int g(x)(y-x)^{\mu_2-1} dx + \\
&\quad \int (x-T)g'(x)(y-x)^{\mu_2-1} dx.
\end{aligned}$$

Using the last equality one sees that

$$\begin{aligned}
(5) &= (\mu_2-1) I_{[\alpha_{r+1}, \alpha_i]}^{\mu_2}(g) + I_{[\alpha_{r+1}, \alpha_i]}^{\mu_2}(g) + \int (x-T)g'(x)(y-x)^{\mu_2-1} dx \\
&= \left(\sum_{k=1}^r b_k \right) I_{[\alpha_{r+1}, \alpha_i]}^{\mu_2}(g)(y),
\end{aligned}$$

where $c_{\mu_1+\mu_2}(\mathbf{a}) = (b_1, \dots, b_r)$ (use that g is a solution of $c_{\mu_1}(D_{\mathbf{a}})$). \square

Remark 6.3 The use of Pochhammer contours is an essential ingredient in the proof of the above lemma (see Formula (5)).

In the following, F denotes a fundamental system of a Fuchsian system $D_{\mathbf{a}}$ and

$$G(x) := \begin{pmatrix} F(x)(x - t_1)^{-1} \\ \vdots \\ F(x)(x - t_r)^{-1} \end{pmatrix}.$$

The next results will be used in the proof of Theorem 6.8:

Lemma 6.4 *i) The columns of G are solutions of $D_{c_{-1}(\mathbf{a})}$.*

$$\text{ii) } I_{[\alpha_{r+1}, \alpha_i]}^\mu(G) = I_{\alpha_i}^\mu(G)(1 - e^{2\pi i \mu}) - I_{\alpha_{r+1}}^\mu(G)(1 - \text{Mon}(\alpha_i)).$$

Proof: The first assertion follows from a straightforward computation. The second assertion follows from the definition of $[\alpha_{r+1}, \alpha_i]$, using the effect of the monodromy on the integrand, see [11], Chap. 18. \square

Corollary 6.5 *i) If μ is a positive integer, then*

$$I_{[\alpha_{r+1}, \alpha_i]}^\mu(G) = 0.$$

ii) If $\mu = 0$ or a negative integer, then

$$I_{[\alpha_{r+1}, \alpha_i]}^\mu(G) = \frac{2\pi i}{-\mu!} G^{(-\mu)}(y)(-1 + \text{Mon}(\alpha_i)).$$

Proof: The claims follow from the above lemma and Cauchy's integral formula. \square

Lemma 6.6 *Let $Y' = \sum \frac{a_i}{x - t_i} Y$ be a Fuchsian system with a nontrivial monodromy group and $\mu \notin \mathbb{Z}$. Then there exists an i and a solution $f(x)$ such that*

$$\int_{[\alpha_{r+1}, \alpha_i]} \frac{f(x)}{x - t_i} (y - x)^{\mu-1} dx \neq 0.$$

Proof: We can assume that we have nontrivial monodromy at $t_1 = 0$. If the monodromy is not unipotent then we can find an entry

$$g(x) = x^\alpha \sum_{j=0}^{\infty} x^j a(j), \quad \alpha \notin \mathbb{Z}, \quad a(0) \neq 0$$

of a solution $f(x)$ near t_1 . Then

$$I_{[\alpha_{r+1}, \alpha_1]}^\mu(g) = \sum_{j=0}^{\infty} a(j) \int_{[\alpha_{r+1}, \alpha_1]} x^{\alpha+j} (y - x)^{\mu-1} dx.$$

Using similar arguments as in [22], Chap. IV, one can prove that

$$\int_{[\alpha_{r+1}, \alpha_1]} x^{\alpha+j} (y-x)^{\mu-1} dx = y^{\alpha+j+\mu} \beta(\alpha+i, \mu),$$

where $\beta(\alpha+i, \mu) \neq 0$, if $\alpha+i, \mu \notin \mathbb{Z}$.

In the case of nontrivial unipotent monodromy at $t_1 = 0$, there exists a solution which has an entry $g(x) = h_0(x) + \log(x)h_1(x)$ near 0, where h_0, h_1 are meromorphic at 0 and $h_1 \neq 0$ (see [11], 16.3). The claim follows now from an easy exercise, using integration by parts, Lemma 6.4 and Corollary 6.5. \square

6.2 The Riemann-Hilbert correspondence for MC_λ

In the notation of the preceding sections. Let $\mathbf{a} := (a_1, \dots, a_r)$, $a_i \in \mathbb{C}^{n \times n}$. Let F be a fundamental matrix of the Fuchsian system $D_{\mathbf{a}} : Y' = \sum \frac{a_i}{x-t_i} Y$ and

$$G(x) := \begin{pmatrix} F(x)(x-t_1)^{-1} \\ \vdots \\ F(x)(x-t_r)^{-1} \end{pmatrix}.$$

Definition 6.7 Let $\mu \in \mathbb{C}$. The matrix

$$I^\mu := I^\mu(y) := (I_{[\alpha_{r+1}, \alpha_1]}^\mu(G)(y), \dots, I_{[\alpha_{r+1}, \alpha_r]}^\mu(G)(y))$$

is called the *period matrix*.

Remark: It follows from the definitions that if the period matrix I^μ is invertible, then it describes the pairing between $H_1(X(y_0), \mathcal{G}_{y_0})$ and $H_{DR}^1(X(y_0), \mathcal{G}_{y_0}^\vee)$, where $X(y_0)$ and \mathcal{G} are as in the proof of Theorem 4.4. In the next theorem we will give criteria for I^μ to be invertible, i.e., the rows of $I^\mu(y_0)$ exhibit a base of $H_{DR}^1(X(y_0), \mathcal{G}_{y_0}^\vee)$.

In a similar way as described in Yoshida [22], Chap. iv, it can be shown that the matrix \mathfrak{H} which occurs in Lemma 2.5 has a natural interpretation as an intersection matrix of (“loaded”) cycles $c \in H_1(X(y_0), \mathcal{G}_{y_0})$.

The next theorem shows the relation between the additive and multiplicative versions of the convolution:

Theorem 6.8 Let $\mathbf{a} := (a_1, \dots, a_r)$, $a_i \in \mathbb{C}^{n \times n}$, $\text{Mon}(D_{\mathbf{a}}) = (A_1, \dots, A_r) \in \text{GL}_n(\mathbb{C})^r$ its tuple of monodromy generators, $\mu \in \mathbb{C} \setminus \mathbb{Z}$ and $\lambda := e^{2\pi i \mu}$. If the generated subgroup $\langle A_1, \dots, A_r \rangle$ is an irreducible subgroup of $\text{GL}_n(\mathbb{C})$ and if at least two elements of A_1, \dots, A_r are $\neq 1$, then the following statements hold:

i) The columns of the period matrix $I^\mu(y)$ are solutions of $D_{c_{\mu-1}(\mathbf{a})}$, where y is contained in a small open neighbourhood U of y_0 .

ii) For $v_i \in \ker(A_i - 1)$, $i = 1, \dots, r$, (resp. $v \in \ker(A_1 \cdots A_r \lambda - 1)$) assume that the residues of $G(x)v_i$ at t_i (resp. the residues of $x^{\mu-1}G(x)v$ at ∞) is not identically zero. Then the period matrix $I^\mu(y)$, $y \in U$, is a fundamental matrix of $D_{c_{\mu-1}(\mathbf{a})}$. Further, the tuple of monodromy generators of $D_{c_{\mu-1}(\mathbf{a})}$ with respect to $I^\mu(y)$ and the paths β_1, \dots, β_r is $C_\lambda(\text{Mon}(D_{\mathbf{a}}))$, i.e.,

$$\text{Mon}(D_{c_{\mu-1}(\mathbf{a})}) = C_\lambda(\text{Mon}(D_{\mathbf{a}})).$$

iii) Assume that

$$\text{rk}(a_i) = \text{rk}(A_i - 1) \quad \text{and} \quad \text{rk}(a_1 + \cdots + a_r + \mu) = \text{rk}(\lambda \cdot A_1 \cdots A_r - 1).$$

The matrix $I^\mu(y)$ gives rise to a fundamental matrix $\tilde{I}^\mu(y)$, $y \in U$, of the system $D_{mc_{\mu-1}(\mathbf{a})}$ (see Remark 5.3, ii). The tuple of monodromy generators of $D_{mc_{\mu-1}(\mathbf{a})}$ with respect to $\tilde{I}^\mu(y)$ and the paths β_1, \dots, β_r is $MC_\lambda(\text{Mon}(D_{\mathbf{a}}))$, i.e.,

$$\text{Mon}(D_{mc_{\mu-1}(\mathbf{a})}) = MC_\lambda(\text{Mon}(D_{\mathbf{a}})).$$

Remark 6.9 a) It follows from the proof that one can weaken the assumptions of Theorem 6.8 such that tuple (A_1, \dots, A_r) fulfills the conditions (*) and (**) of Subsection 2.2 instead of the irreducibility and non-triviality condition on A_1, \dots, A_r .

b) In Theorem 6.8, iii), if $\text{rk}(a_i) > \text{rk}(A_i - 1)$ then the differential system, which corresponds to (the local system corresponding to) $MC_\lambda(A_1, \dots, A_r)$ is a factor system of $D_{mc_{\mu-1}(\mathbf{a})}$.

Proof of i): This follows from Lemma 6.2 and Lemma 6.4 i). \square

Proof of ii): Let V_λ denote the $\pi_1(X \setminus \{y_0\}, x_0)$ -module whose underlying vector space is the column vector space \mathbb{C}^n on which α_i acts via A_i ($1 \leq i \leq r$) and α_{r+1} acts via λ . Let \mathbb{C}_n denote the space of row vectors and V_λ^\vee the dual module of V_λ with respect to $\mathbb{C}_n \times \mathbb{C}^n \rightarrow \mathbb{C}$, $(w, v) \mapsto w \cdot v$. Let F^j denote the j -th row of F and

$$\delta_{i,j} : \pi_1(X \setminus \{y_0\}, x_0) \rightarrow V_\lambda^\vee, \gamma \mapsto \int_\gamma F^j (y_0 - x)^{\mu-1} \frac{dx}{x - t_i}.$$

By the properties of the integral, $\delta_{i,j}$ is an element in $C^1(\pi_1(X \setminus \{y_0\}, x_0), V_\lambda^\vee)$. It follows that the rows of $I^\mu(y_0)$ are exactly the images of the crossed homomorphisms $\delta_{i,j}$ under the twisted evaluation map. By the definition of $I_{[\alpha_{r+1}, \alpha_i]}^\mu$, analytic continuation of $I^\mu(y)$ along the path β_k transforms $I_{[\alpha_{r+1}, \alpha_i]}^\mu(G)$ into $I_{[\beta_k \alpha_{r+1}, \beta_k \alpha_i]}^\mu(G)$. It follows then from Theorem 3.3 (Formula (2)), that the matrix which describes this transformation is the matrix B_k , where

$$C_\lambda(A_1, \dots, A_r) = (B_1, \dots, B_r).$$

In order to prove ii), it remains to prove that the columns of $I^\mu(y)$ form a fundamental set of solutions. This follows from the Lemmata below:

Consider the vector space of solutions $J := I^\mu(y) \cdot \mathbb{C}^{nr}$, with y in a small neighborhood of y_0 . Let further \mathcal{K}_i , \mathcal{K} and \mathcal{L} be as in Subsection 2.1 and $\hat{\mathcal{K}}_i := I^\mu(y) \cdot \mathcal{K}_i$, $\hat{\mathcal{K}} := I^\mu(y) \cdot \mathcal{K}$ and $\hat{\mathcal{L}} := I^\mu(y) \cdot \mathcal{L}$.

Lemma 6.10 *The kernel of the map*

$$I^\mu : \mathbb{C}^{nr} \rightarrow J, (v_1, \dots, v_r)^{\text{tr}} \rightarrow I^\mu(y) \cdot (v_1, \dots, v_r)^{\text{tr}}$$

is a $\langle B_k : k = 1, \dots, r \rangle$ -module.

Proof: If $I^\mu v = 0$, then $I^\mu B_k v = 0$. □

If $G = (g_{i,j}(t))$ is a vector valued function which is componentwise meromorphic at t_k , then $\text{Res}_{t_k}(G)$ denotes the vector of residues $(\text{Res}_{t_k}(g_{i,j}(t)))$.

Lemma 6.11 *Let $\mu \notin \mathbb{Z}$. Then the functions in $\hat{\mathcal{K}}_i$ (resp. $\hat{\mathcal{L}}$) have at most a singularity at t_i (resp. ∞). Moreover,*

$$\text{i)} \quad \hat{\mathcal{K}}_i = \langle \text{Res}_{t_i}(G(x)v)(y - t_i)^{\mu-1} \mid v \in \ker(A_i - 1) \rangle$$

for $i = 1, \dots, r$.

$$\text{ii)} \quad \hat{\mathcal{L}} = \langle \text{Res}_\infty(x^{\mu-1}G(x)v) \mid v \in \ker(A_1 \cdots A_r \lambda - 1) \rangle$$

Proof: i) One has

$$\begin{aligned} \hat{\mathcal{K}}_i = I^\mu \mathcal{K}_i &= I_{[\alpha_{r+1}, \alpha_i]}^\mu(G(x)) \ker(A_i - 1) \\ &= I_{\alpha_i}^\mu(G(x)) \ker(A_i - 1) \end{aligned}$$

by Lemma 6.4. The claim follows from Cauchy's integral formula since $G(x)v$ (as matrix valued function in x) is meromorphic at t_i for $v \in \ker(A_i - 1)$.

ii) Using Lemma 6.4 one easily sees that

$$\hat{\mathcal{L}} = I_{[\alpha_{r+1}, \alpha_\infty]}^\mu(G(x)) \ker(A_1 \cdots A_r \lambda - 1),$$

where $\alpha_\infty = \alpha_1 \cdots \alpha_{r+1}$. Using the same arguments as in i) the claim follows. □

Corollary 6.12 *One has*

$$\hat{\mathcal{K}} + \hat{\mathcal{L}} = \oplus_i \hat{\mathcal{K}}_i \oplus \hat{\mathcal{L}}$$

as a left- $\langle B_1, \dots, B_r \rangle$ -module.

Lemma 6.13 *If the conditions (*) and (**) of Subsection 2.2 hold for $\text{Mon}(D_{\mathbf{a}}) = (A_1, \dots, A_r)$, then*

$$\ker(I^\mu) \leq \mathcal{K} + \mathcal{L}.$$

Proof: Assume that $\ker(I^\mu) \not\subseteq \mathcal{K} + \mathcal{L}$. Let $O \leq V_1 \leq \dots \leq V_k = V_\lambda$ be a composition series of V_λ (as a module). Let further V_i^r be the corresponding (diagonal) subspace of $V_\lambda^r = \mathbb{C}^{nr}$ and $\tilde{V}_i := V_i^r + \mathcal{K} + \mathcal{L} \pmod{\mathcal{K} + \mathcal{L}}$. It follows from Theorem 2.4, iii), and [9], Lemma 2.8, that $O \leq \tilde{V}_1 \leq \dots \leq \tilde{V}_k = V_\lambda^r / (\mathcal{K} + \mathcal{L})$ is a composition series of $V_\lambda^r / (\mathcal{K} + \mathcal{L})$ (as $\langle B_1, \dots, B_r \rangle$ -module). Since $\ker(I^\mu)$ is a $\langle B_1, \dots, B_r \rangle$ -module, there exists a $\langle A_1, \dots, A_r \rangle$ -module $W \leq V_\lambda$ such that $W^r + \mathcal{K} + \mathcal{L} \leq \ker(I^\mu) + \mathcal{K} + \mathcal{L}$. We assume that W is minimal and nontrivial. By minimality, (*) and (**), also hold for W , see [9], proof of Corollary 3.6.

Property (**) for W implies that

$$(6) \quad I^\mu(y)(B_i - 1)W^r = I^\mu(y) \begin{pmatrix} 0 \\ \vdots \\ W \\ \vdots \\ 0 \end{pmatrix} \quad (i\text{-th entry}).$$

By assumption on W , one has

$$I^\mu(y) \begin{pmatrix} w_1 \\ \vdots \\ w_r \end{pmatrix} = \left(\sum_{k=1}^r g_k \right) + g_\infty,$$

where $w_1, \dots, w_r \in W$ and $g_i \in \hat{\mathcal{K}}_i$ (resp. $g_\infty \in \hat{\mathcal{L}}$), by Lemma 6.11. Using the monodromy around t_i we get

$$I^\mu(y)B_i \begin{pmatrix} w_1 \\ \vdots \\ w_r \end{pmatrix} = \left(\sum_{k \neq i} g_k \right) + g_\infty + \lambda g_i.$$

Subtracting these equalities one obtains $I^\mu(y)(B_i - 1)W^r \leq \hat{\mathcal{K}}_i$ and (together with Equation (6))

$$I^\mu(y)(B_i - 1)W^r = I^\mu(y) \begin{pmatrix} 0 \\ \vdots \\ W \\ \vdots \\ 0 \end{pmatrix} \leq \hat{\mathcal{K}}_i.$$

Using the description of $\hat{\mathcal{K}}_i$ in terms of functions, one sees that for $j = 1, \dots, r, j \neq i$,

$$I^\mu(y)(B_j - 1) \begin{pmatrix} 0 \\ \vdots \\ W \\ \vdots \\ 0 \end{pmatrix} = I^\mu(y) \begin{pmatrix} 0 \\ \vdots \\ (A_i - 1)W \\ \vdots \\ 0 \end{pmatrix} = 0,$$

where the expression on the right hand side of the first equality is zero outside the j -th block entry. Similarly one obtains

$$I^\mu(y)(B_i - \lambda) \begin{pmatrix} 0 \\ \vdots \\ W \\ \vdots \\ 0 \end{pmatrix} = I^\mu(y) \begin{pmatrix} 0 \\ \vdots \\ (A_i - 1)W \\ \vdots \\ 0 \end{pmatrix} = 0.$$

Since (**) holds for W , a block-wise argument shows that

$$(7) \quad W^r \leq \ker(I^\mu).$$

On the other hand, since $\bigcap_{i=1}^r \ker(A_i|_W - 1) = 0$ (Property (*)), we can find an $i \in \{1, \dots, r\}$ and a solution f in $F \cdot W$ (where F is a fundamental system of $D_{\mathbf{a}}$), such that f has nontrivial monodromy at t_i . The Euler transform

$$g := I_{[\alpha_{r+1}, \alpha_i]}^\mu \left(\left(\frac{f}{x-t_1}, \dots, \frac{f}{x-t_r} \right)^{\text{tr}} \right)$$

is a solution of $D_{c_{\mu-1}(\mathbf{a})}$. Lemma 6.6 implies then that g is not identically zero. This gives a contradiction to Equation (7), so $W = 0$ and the claim follows. \square

Finish of the proof of ii): It follows from the assumptions on the residues and Lemmata 6.11 and 6.12 that $\dim(\hat{\mathcal{K}}) = \dim(\mathcal{K})$ and $\dim(\hat{\mathcal{L}}) = \dim(\mathcal{L})$. It follows then from Lemma 6.13 that the columns of $I^\mu(y)$ are linearly independent. \square

Proof of iii): This follows from dimension reasons (using the rank-conditions) and Lemma 6.13. \square

7 Applications of the convolution functors MC_λ and mc_μ

7.1 Rigid local systems and Fuchsian systems

In this subsection we want to outline a construction algorithm for Fuchsian systems corresponding to irreducible rigid local systems under the Riemann-Hilbert correspondence.

For $\Omega = (\omega_1, \dots, \omega_r) \in (K^\times)^r$, the *scalar multiplication with Ω*

$$\text{GL}_n(K)^r \rightarrow \text{GL}_n(K)^r, (A_1, \dots, A_r) \mapsto (\omega_1 A_1, \dots, \omega_r A_r)$$

is denoted by M_Ω . The corresponding effect on local systems on the r -punctured affine line is also denoted by M_Ω . Similarly, for $\Delta = (\delta_1, \dots, \delta_r) \in K^r$, the *scalar addition with Δ*

$$(K^{n \times n})^r \rightarrow (K^{n \times n})^r, (a_1, \dots, a_r) \mapsto (a_1 + \delta_1 \cdot 1, \dots, a_r + \delta_r \cdot 1)$$

is denoted by m_Δ . The corresponding effect on Fuchsian systems is also denoted by m_Δ .

Let \mathcal{F} be a complex irreducible (physically) rigid local system. By the results of [12], Chap. 6, and [9], Chap. 4, one can construct \mathcal{F} by applying iteratively a suitable sequence of scalar multiplications M_{Ω^i} (see [9], Chap. 4) and middle convolutions MC_{λ_j} to a one-dimensional local system \mathcal{F}_0 .

It is easy to write down a Fuchsian system

$$D_{\mathbf{a}^0} : Y' = \left(\frac{a_1^0}{x-t_1} + \cdots + \frac{a_r^0}{x-t_r} \right) Y, \quad a_i^0 \in \mathbb{C},$$

whose solutions form the local system \mathcal{F}_0 . This system is irreducible and rigid, and we (can) assume that it fulfills the assumptions of Theorem 6.8 iii) (i.e., if $a_i^0 \in \mathbb{Z}$ then $a_i^0 = 0$, and there exist at least two elements $a_{i_1}^0, a_{i_2}^0$ such that $a_{i_1}^0, a_{i_2}^0 \notin \mathbb{Z}$). It follows now from Theorem 6.8 iii) that there exists a sequence of scalar additions

$$m_{\Delta^i}, \Delta^i = (\delta_1^i, \dots, \delta_r^i), \quad \text{such that} \quad (e^{2\pi i \delta_1^i}, \dots, e^{2\pi i \delta_r^i}) = \Omega^i,$$

and middle convolutions

$$mc_{\mu_j}, e^{2\pi i \mu_j} = \lambda_j,$$

such that the iterative application of this sequence to $D_{\mathbf{a}^0}$ yields an irreducible Fuchsian system D whose monodromy coincides with the monodromy of \mathcal{F} . The only thing one has to take care of, is to choose the scalar additions (modulo \mathbb{Z}) that the rank condition of Theorem 6.8 iii) is fulfilled. This is possible in every step by the following argument:

It is shown in [9], Chap. 4, resp. Appendix A, that, in the irreducible case, the index of rigidity is preserved by MC_λ , resp. mc_μ (which is equal to 2 in both, the additive and the multiplicative, cases). By the compatibility between MC_λ and mc_μ (dimension reasons), one can see that if two eigenvalues of a matrix which occurs as a component in one step of the above ‘‘additive’’ construction differ by an element of \mathbb{Z} , then they correspond to a certain Jordan block of length > 1 in a matrix which occurs in the ‘‘multiplicative’’ construction, in a way that the rank condition of Theorem 6.8 iii) is fulfilled.

By the construction of mc_μ , it clear that everything can be done in an algorithmic way and is easily implemented on the computer. Moreover, one obtains the sections of the local system \mathcal{F} in a concrete way as iterated integrals, compare to [10] and [21].

Remark: By Crawley-Boevey’s solution of the additive Deligne-Simpson problem (see [7]) the rigid tuples of complex matrices having sum $\equiv 0$ are known and, by the additive Katz’ existence algorithm (see [9], Appendix A), these tuples can be constructed similar to the above construction.

But in general, it is a difficult problem to decide when the associated Fuchsian system is irreducible, i.e., the associated local system of solutions is an

irreducible rigid local system. The point is, that in the above construction, the irreducibility is ensured by Theorem 6.8 iii), using the fact that (under the given assumptions) the functors MC_{λ_j} preserve irreducibility, see Theorem 2.4, iii).

7.2 Geometric differential equations

Let X be a smooth and geometrically connected algebraic variety over an algebraically closed field $K \subseteq \mathbb{C}$, $f : Y \rightarrow X$ a smooth projective morphism and d the universal differential $Y \rightarrow \Omega_Y^1$. The Gauß-Manin connection on relative de Rham cohomology groups $H_{\text{DR}}^i(Y/X) := R^i f_*^{\text{DR}}(\mathcal{O}_Y, d)$ gives rise to a system of differential equations (see [1] for details). A differential system is said to be *arising from geometry* if it is equivalent to an iterated extension of subfactors of such differential systems (see [1], Chap. II).

Theorem 7.1 *Let K be a number field, $\mathbf{a} = (a_1, \dots, a_r)$, $a_i \in K^{n \times n}$, $\mu \in \mathbb{Q}$, such that the conditions of Theorem 6.8 hold for $D_{\mathbf{a}}$. If $D_{\mathbf{a}}$ is arising from geometry, then $D_{c_\mu(\mathbf{a})}$ (resp. $D_{mc_\mu(\mathbf{a})}$) is arising from geometry.*

Proof: The claim follows from the construction of the period matrix and the result of André [1], saying that the category of differential modules which arise from geometry is closed under taking higher direct images. \square

Let us consider an example:

Lemma 7.2 *Let*

$$p(x) = 4(x - t_1)(x - t_2)(x - t_3) = 4x^3 - g_2x - g_3,$$

$B \in \mathbb{C}$ and

$$L_n := L_n(p, B) := p(x)y'' + \frac{1}{2}p'(x)y' - (n(n+1)x + B)y$$

the Lamé differential equation of index $n \in \mathbb{Q}$. Then, L_n can be transformed into the Fuchsian system

$$\begin{aligned} Y' &= \sum_{i=1}^3 \frac{a_i}{x - t_i} Y \\ &:= \left(\frac{1}{x - t_1} \begin{pmatrix} 0 & 1 \\ 0 & \frac{1}{2} \end{pmatrix} + \frac{1}{x - t_2} \begin{pmatrix} 0 & 0 \\ l_1 & -\frac{1}{2} \end{pmatrix} + \frac{1}{x - t_3} \begin{pmatrix} 0 & 0 \\ l_2 & -\frac{1}{2} \end{pmatrix} \right) Y, \end{aligned}$$

where

$$l_1 = \frac{t_2 n(n+1) + B}{4(t_2 - t_3)} \quad \text{and} \quad l_1 + l_2 = \frac{n(n+1)}{4}.$$

Proof: The differential system which corresponds to L_n is

$$Y' = \begin{pmatrix} 0 & 1 \\ \frac{n(n+1)x+B}{p(x)} & -\frac{1}{2} \frac{p'(x)}{p(x)} \end{pmatrix} Y.$$

Using the gauge transformation $Y \mapsto \begin{pmatrix} 1 & 0 \\ 0 & x-t_1 \end{pmatrix} Y$ we get the equivalent system

$$Y' = \left(\frac{1}{x-t_1} \begin{pmatrix} 0 & 1 \\ 0 & \frac{1}{2} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \frac{n(n+1)x+B}{4(x-t_2)(x-t_3)} & -\frac{1}{2} \sum_{i=2}^3 \frac{1}{x-t_i} \end{pmatrix} \right) Y.$$

Since

$$\frac{n(n+1)x+B}{4(x-t_2)(x-t_3)} = \frac{t_2 n(n+1)+B}{4(x-t_2)(t_2-t_3)} - \frac{t_3 n(n+1)+B}{4(x-t_3)(t_2-t_3)},$$

the claim follows. \square

Definition 7.3 We say that a system of differential equations D is in *Okubo normal form*, if

$$D : Y' = (x-T)^{-1} b Y \quad \text{or, equivalently,} \quad D : (x-T)Y' = bY,$$

where $b \in \mathbb{C}^{n \times n}$ and T is a diagonal matrix $T = \text{diag}(t_1, \dots, t_n)$, $t_i \in \mathbb{C}$ (here possibly $t_i = t_j$ for $i \neq j$).

Lemma 7.4 Let $r \geq 3$, $\mu \in \mathbb{C} \setminus \mathbb{Z}$ and $\mathbf{a} := (a_1, \dots, a_r)$, $a_i \in \mathbb{C}^{2 \times 2}$, where a_1, a_2, a_3 are as in the previous lemma. If $\text{rk}(a_i) = 2$ for $i > 3$, and if $-\mu$ is no eigenvalue of $a_1 + \dots + a_r$, then $D_{mc_\mu(\mathbf{a})}$ is equivalent to the following differential system in Okubo form:

$$D(L_n, \mathbf{a}, \mu) : (x-T)Y' = (\tilde{c} + \mu)Y$$

where $T = \text{diag}(t_1, t_2, t_3, t_4, t_4, \dots, t_r, t_r)$ and

$$\tilde{c} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & (0,1)a_4 & \dots & (0,1)a_r \\ -2l_1 + \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & (-2l_1, 1)a_4 & \dots & (-2l_1, 1)a_r \\ -2l_2 + \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & (-2l_2, 1)a_4 & \dots & (-2l_2, 1)a_r \\ \hline 1 & 0 & 0 & a_4 & \dots & a_r \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & & & \\ \hline \vdots & \vdots & \vdots & \vdots & & \vdots \\ \hline 1 & 0 & 0 & a_4 & \dots & a_r \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & & & \end{pmatrix}$$

Proof: In the notation of Subsection 5.1. Let $c_0(\mathbf{a}) = (b_1, \dots, b_r)$, where $b_i \in \mathbb{C}^{2r \times 2r}$. One has $D_{c_\mu(\mathbf{a})} : Y' = (x - T)^{-1}(b + \mu)Y$, where $b := b_1 + \dots + b_r$.

Let

$$B_1 := \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}, B_2 := \begin{pmatrix} 1 & 0 \\ -2l_1 & 1 \end{pmatrix}, B_3 := \begin{pmatrix} 1 & 0 \\ -2l_2 & 1 \end{pmatrix}.$$

Then

$$B_1 a_1 B_1^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \quad \text{and} \quad B_2 a_2 B_2^{-1} = B_3 a_3 B_3^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}.$$

Consider the block-diagonal matrix $d := \text{diag}(B_1, B_2, B_3, E_2, \dots, E_2)$. One computes that

$$dbd^{-1} = \left(\begin{array}{ccc|ccc} B_1 a_1 B_1^{-1} & B_1 a_2 B_2^{-1} & B_1 a_3 B_3^{-1} & B_1 a_4 & \dots & B_1 a_r \\ B_2 a_1 B_1^{-1} & B_2 a_2 B_2^{-1} & B_2 a_3 B_3^{-1} & B_2 a_4 & \dots & B_2 a_r \\ B_3 a_1 B_1^{-1} & B_3 a_2 B_2^{-1} & B_3 a_3 B_3^{-1} & B_3 a_4 & \dots & B_3 a_r \\ \hline a_1 B_1^{-1} & a_2 B_2^{-1} & a_3 B_3^{-1} & a_4 & \dots & a_r \\ \vdots & \vdots & \vdots & & \vdots & \\ a_1 B_1^{-1} & a_2 B_2^{-1} & a_3 B_3^{-1} & a_4 & \dots & a_r \end{array} \right)$$

It is easily checked that conjugating b with d gives an equivalence between $D_{c_\mu(\mathbf{a})}$ and $Y' = (x - T)^{-1}(c + \mu)Y$ with $c = dbd^{-1}$.

Under the assumptions, the space $\mathfrak{l} \leq \mathbb{C}^{2r}$ is zero. Since $B_i a_i B_i^{-1}$ is diagonal, factoring out the space \mathfrak{k} corresponds to canceling the first, third and fifth row and column of c . This yields \tilde{c} . \square

Baldassarri [4] gives examples of Lamé equations with finite monodromy. E.g., it is shown that the monodromy group of $L_n(p(x), B)$, ($n = 1$, $p(x) = 4x^3 + g_3$, $B = 0$), is the symmetric group on 3 letters. Beukers and van der Waall [5] list all finite groups which can occur as the monodromy groups of Lamé equations and they give many examples of such equations. Also, van der Waall [18] gives an algorithm to detect all Lamé equations with finite monodromy group. One equation which can be found in [5], Table 4, is given by $L_n(p(x), B)$ ($n = 1/6$, $p(x) = 4x^3 - x$, $B = 0$), whose monodromy group is isomorphic to the complex reflection group G_{13} .

This yields a new family of differential systems arising from geometry:

Corollary 7.5 *Under the assumptions of Lemma 7.4. Let $L_n(p, B)$ a Lamé equation which has finite monodromy (e.g., if $n = 1$, $p(x) = 4x^3 + g_3$, $B = 0$, or $n = 1/6$, $p(x) = 4x^3 - x$, $B = 0$), a_4, \dots, a_r scalar matrices contained in $\mathbb{Q}^{2 \times 2}$ and $\mu \in \mathbb{Q}$. Then the following holds:*

- i) *The differential system $D(L_n(p, B), \mathbf{a}, \mu)$ is arising from geometry.*
- ii) *The solutions of $D(L_n(p, B), \mathbf{a}, \mu)$ are G -functions.*

Proof: This follows from Theorem 7.1 and [1], Chap. V. \square

7.3 Transformation of the p -curvature under mc_μ

In this section, we want to study how the p -curvature changes under the convolution:

Let K be a number field and $D : Y' = aY$, where $a = (a_{i,j}) \in K(x)^{n \times n}$. Successive application of differentiation yields differential systems

$$D^{(n)} : Y^{(n)} = \widehat{a^{(n)}}Y.$$

In the following, \mathfrak{p} always denotes a prime of K which lies over p . For almost all primes p , one can reduce $\widehat{a^{(n)}}$ modulo \mathfrak{p} , in order to obtain the p -curvature matrices

$$a(\mathfrak{p}) := \widehat{a^{(n)}} \pmod{\mathfrak{p}}.$$

The p -curvature matrices encode many arithmetic properties of the differential system D and are conjecturally related to questions about the geometric nature of D :

Conjecture 7.6 *i) (Grothendieck-Katz, see [13], [2]) The Lie algebra of the differential Galois group of D is minimal to the property that, for almost all primes \mathfrak{p} of K , its reduction modulo \mathfrak{p} contains the p -curvature matrix $a(\mathfrak{p})$.*

ii) (Bombieri-Dwork, see [1]) If D is globally nilpotent, i.e., $a(\mathfrak{p})$ is nilpotent for almost all primes \mathfrak{p} , then D is arising from geometry.

Remark: (i) The Grothendieck-Katz conjecture implies the p -curvature conjecture of Grothendieck: If $a(\mathfrak{p}) = 0$ for almost all \mathfrak{p} , then D has a fundamental set of solutions consisting of algebraic functions.

(ii) It is well known that if D has a fundamental set of solutions consisting of algebraic functions, then $a(\mathfrak{p}) = 0$ for almost all \mathfrak{p} . Also, if D is arising from geometry, then $a(\mathfrak{p})$ is nilpotent for almost all \mathfrak{p} .

Remark 7.7 (Okubo) Let $D : (x - T)Y' = bY$ be a system of differential equations in Okubo normal form. Then, $(x - T)Y^{(2)} = (b - 1)Y'$.

An induction yields the following recursion formula for the p -curvature matrix of a system of differential equations in Okubo normal:

Lemma 7.8 *Let $D : (x - T)Y' = bY$ be a system of differential equations in Okubo normal form. Then*

$$\widehat{a^{(n)}} = (x - T)^{-1}(b - n + 1) \cdot (x - T)^{-1}(b - n + 2) \cdots (x - T)^{-1}(b - 1) \cdot (x - T)^{-1}b.$$

Theorem 7.9 *Let $\mathbf{a} = (a_1, \dots, a_r)$, $a_i \in K^{n \times n}$, such that*

$$a(\mathfrak{p})^k = 0.$$

Let $\mu \in \mathbb{Q}$ and denote by $c_\mu(a(\mathbf{p}))$ (resp. $mc_\mu(a(\mathbf{p}))$) the p -curvature matrix of $D_{c_\mu(\mathbf{a})}$ (resp. $D_{mc_\mu(\mathbf{a})}$).

i) If $\mu = -1$, then $c_\mu(a(\mathbf{p}))^{k+1} = 0$ and $mc_\mu(a(\mathbf{p}))^{k+1} = 0$.

ii) If $\mu = \frac{n_1}{n_2}$ and p does not divide $n_1 n_2$, then

$$c_{\mu-1}(a(\mathbf{p}))^{k+2} = 0 \text{ and } mc_{\mu-1}(a(\mathbf{p}))^{k+2} = 0.$$

Proof: The convolution of $D_{\mathbf{a}}$ is a differential system in Okubo normal form:

$$D_{c_\mu(\mathbf{a})} : Y' = \sum_{k=1}^r \frac{b_k}{x-t_k} Y = (x-T)^{-1} \left(\sum_{k=1}^r b_k \right) Y,$$

where T is the diagonal matrix $T = \text{diag}(t_1, \dots, t_1, \dots, t_r, \dots, t_r)$ (every t_k occurs n times) and b_k is as in Section 5.1. If $\mu = -1$ then $(\sum_{k=1}^r b_k)$ is a blockmatrix $b = (b_{i,j})$ with $b_{i,j} = a_j - \delta_{i,j} E_n$.

Using the gauge transformations with $(x-T)$ and

$$H := \begin{pmatrix} E_n & -E_n & 0 & \dots \\ 0 & \ddots & \ddots & \\ \vdots & & & -E_n \\ 0 & \dots & & E_n \end{pmatrix}$$

one sees that $D_{c_{-1}(\mathbf{a})}$ is equivalent to the following system:

$$Y' = \begin{pmatrix} 0 & \dots & \dots & 0 \\ \vdots & & & \vdots \\ 0 & \dots & \dots & 0 \\ \frac{a_1}{x-t_1} & (\frac{a_1}{x-t_1} + \frac{a_2}{x-t_2}) & \dots & (\frac{a_1}{x-t_1} + \dots + \frac{a_r}{x-t_r}) \end{pmatrix} Y.$$

Thus $c_{-1}(a(\mathbf{p}))$ is equivalent to

$$\begin{pmatrix} 0 & \dots & \dots & 0 \\ \vdots & & & \vdots \\ 0 & \dots & \dots & 0 \\ * & * & * & a(\mathbf{p}) \end{pmatrix}.$$

It follows from $a(\mathbf{p})^k = 0$ that $c_{-1}(a(\mathbf{p}))^{k+1} = 0$ and i) follows.

Let $\hat{\mu} \in \mathbb{N}^+$ be the smallest natural number such that $\hat{\mu} \equiv \mu \pmod{\mathfrak{p}}$ and $b_\infty := b_1 + \dots + b_r - \mu$, where $c_\mu(\mathbf{a}) = (b_1, \dots, b_r)$. Let further

$$h_1 := (x-T)^{-1} b_\infty \cdot (x-T)^{-1} (b_\infty + 1) \cdots (x-T)^{-1} (b_\infty - 1 + \hat{\mu})$$

and

$$h_2 := (x-T)^{-1} (b_\infty + \hat{\mu}) \cdot (x-T)^{-1} (b_\infty + \hat{\mu} + 1) \cdots (x-T)^{-1} (b_\infty - 1 + p).$$

Then $c_{\mu-1}(a(\mathfrak{p})) = h_2 h_1 \pmod{\mathfrak{p}}$ and $c_{-1}(a(\mathfrak{p})) = (h_1 h_2 \pmod{\mathfrak{p}})$ by the above recursion formula (Lemma 7.8). It follows from

$$c_{-1}(a(\mathfrak{p}))^{k+1} = ((h_1 h_2)^{k+1} \pmod{\mathfrak{p}}) = 0,$$

that

$$c_{\mu-1}(a(\mathfrak{p}))^{k+2} = ((h_2 h_1)^{k+2} \pmod{\mathfrak{p}}) = 0,$$

giving ii). □

Corollary 7.10 *Let $\mathbf{a} = (a_1, \dots, a_r)$, $a_i \in K^{n \times n}$, such that $D_{\mathbf{a}}$ is a globally nilpotent and $\mu \in \mathbb{Q}$. Then $D_{c_{\mu}(\mathbf{a})}$ (resp. $D_{mc_{\mu}(\mathbf{a})}$) is globally nilpotent.*

Lemma 7.11 *Let $D(L_n(p(x), B), \mathbf{a}, \mu)$ be as in Corollary 7.5.*

- i) *The Grothendieck-Katz conjecture is true for $D(L_n, \mathbf{a}, \mu)$.*
- ii) *The system $D(L_n, \mathbf{a}, \mu)$ is globally nilpotent of rank 3.*

Proof: i) In the notation of Section 4. Let G be the monodromy group of $D_{\mathbf{a}}$. Let $X = \mathbb{C} \setminus T$ (remember that t_1, t_2, t_3 are determined by L_n), $X_1 \rightarrow X$ be the unramified cover of X which is associated to the homomorphism $\pi_1(X) \rightarrow G \leq \mathrm{GL}_2(\mathbb{C})$ and X_2 the cyclic cover of \mathbb{C}^\times which is associated to $\pi_1(\mathbb{C}^\times) \rightarrow \mathbb{C}^\times$, $\gamma \mapsto e^{2\pi i \mu}$. Let $Y_1 := X_1 \times_X E$, $Y_2 := X_2 \times_{\mathbb{C}^\times} E$ and $\tilde{Y} := Y_1 \times_E Y_2$. By construction, \tilde{Y} is an unramified cover of E and admits, via p_2 , a map $\tilde{f} : \tilde{Y} \rightarrow X$. Let Y denote the compactification of \tilde{Y} with respect to the first coordinate and $f : Y \rightarrow X$ the morphism induced by \tilde{f} . It follows from Riemann's existence theorem that f arises from an underlying smooth map of varieties (i.e., f is the effect on the complex points), denoted by $\hat{f} : \hat{Y} \rightarrow \hat{X}$, where \hat{X} and \hat{Y} are smooth connected varieties over some number field K . It follows from the Leray spectral sequence that $D_{mc_{\mu}(\mathbf{a})}$ is a differential system which is equivalent to a subfactor of the Gauß-Manin connection $\nabla : H_{\mathrm{DR}}^1(\hat{Y}/\hat{X}) \rightarrow \Omega_{\hat{X}}^1 \otimes H_{\mathrm{DR}}^1(\hat{Y}/\hat{X})$.

By [3], Theorem 0.7.1, the claim follows from the connectivity of the motivic Galois group of at least one fibre \hat{Y}_s (which is a nonsingular curve in our case), where s is a geometric point of \hat{X} . But this follows analogously to [3], Ex. 16.3, from the results of [2], relating the motivic Galois group of \hat{Y}_s to the Mumford-Tate group of the Jacobian of \hat{Y}_s .

The claim ii) follows from Theorem 7.9, ii). □

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